Duke preprint DU–TH–143

Gluon Radiation and Coherent States in Ultrarelativistic Nuclear Collisions

Sergei G. Matinyan[1,2], Berndt Müller[1], and Dirk H. Rischke[1]

[1] *Department of Physics, Duke University*
Box 90305, Durham, North Carolina 27708-0305

[2] *Yerevan Physics Institute, Yerevan 375036, Armenia*

We explore the correspondence between classical gluon radiation and quantum radiation in a coherent state for gluons produced in ultrarelativistic nuclear collisions. The expectation value of the invariant momentum distribution of gluons in the coherent state is found to agree with the gluon number distribution obtained classically from the solution of the Yang–Mills equations. A criterion for the applicability of the coherent state formalism to the problem of radiation in ultrarelativistic nucleus–nucleus collisions is discussed. This criterion is found to be fulfilled for midrapidity gluons with perturbative transverse momenta larger than $\sim 1-2$ GeV and produced in collisions between valence partons.

PACS number(s): 12.38.Bx, 12.38.Aw, 24.85.+p, 25.75-q

I. INTRODUCTION

The planned experiments at Brookhaven and CERN aimed at producing a dense quark–gluon plasma [1] in highly energetic nucleus–nucleus collisions [2] have generated interest in theoretical descriptions of the formation of such a state. One approach, the parton cascade model [3], describes the equilibration process as a sequence of binary collisions of perturbative quanta in quantum chromodynamics (QCD). An unresolved issue of this model is how the decoherence of the initial nuclear parton distributions occurs. Two mechanisms have been proposed: the formation of a large number of mini-jets in the nuclear collision [4–6] and the radiation of soft gluons from the colliding valence quarks [7]. We are here concerned with the second mechanism. We note that it was recently argued in [8] that the two approaches may be intimately related and lead to almost identical predictions in certain limits.

The gluon radiation mechanism was originally [7] formulated in the framework of the classical approach proposed by McLerran and Venugopalan [9]. In their model, the small–x gluon distribution of a large nucleus is described as arising from the classical gauge field created by a random distribution of color sources representing the valence quarks of the nucleons and moving along the light-cone, and by the quantum fluctuations around this field. Kovchegov [10] has shown how the classical field can be obtained from a model of the fast moving nucleus as a collection of color dipoles and has identified the limitations of the classical approximation [11].

When two nuclei collide, a part of their classical fields is scattered on-shell, describing the emission of real gluons [7]. A detailed derivation of this process has recently been given to lowest and next-to-lowest order in the QCD coupling constant g [12]. Using semi-classical arguments, the radiated field energy can be related to the gluon multiplicity distribution as function of rapidity and transverse momentum. The result was shown to agree with the prediction of the perturbative, lowest order quantum calculation of soft gluon emission by colliding color charges [13].

In this work we demonstrate that classical gluon radiation as computed in [7,12] is closely related to quantum radiation of gluons in a coherent state, and in fact gives the same result for the gluon number distribution. In the coherent state, gluons are emitted in independent binary interactions between the colliding valence quarks of the two nuclei. Thus, if the number of emitted gluons is only a small fraction of the number of colliding color charges, correlations in the emission of individual gluons, such as arising from multiple collisions of the same charge, are negligible and the coherent state approach agrees with a quantum calculation of the multiple scattering process. If, on the other hand, a significant fraction of the colliding charges radiates, such correlations could become important and corrections to the coherent state calculation have to be accounted for. On the basis of this argument we estimate that for central $Au + Au$–collisions at RHIC energies the coherent state approach is applicable to describe gluon radiation, if the gluons are produced in collisions between valence partons and at midrapidity with perturbative transverse momenta above about $1 - 2$ GeV.

The remainder of this paper is organized as follows. In Section II we briefly outline the results of [12] as far as they are of relevance to this work. Section III consists of two parts. In the first, the coherent state corresponding to gluon radiation from collisions of classical color charges is discussed. The second part interprets the results in terms of quantum diagrams. Conditions for the validity of the coherent state approach are discussed. Section IV contains the derivation of the event-averaged gluon number distribution in the coherent state formalism. The result is found to agree with that of Ref. [12]. We then generalize it to nuclear collisions at finite impact parameter. Section V concludes this work with a numerical evaluation of the gluon number distribution and a quantitative discussion of the validity of the coherent state approach for the description of gluon radiation in ultrarelativistic nuclear collisions.

Our units are $\hbar = c = 1$, and the metric tensor is $g^{\mu\nu} = \mathrm{diag}(+,-,-,-)$. Light-cone coordinates are defined as $a_\pm \equiv (a^0 \pm a^z)/\sqrt{2}$, $\partial_\mp \equiv \partial/\partial x_\pm$. The notation for 3–vectors is $\mathbf{a} = (a^x, a^y, a^z)$ and for transverse vectors $\underline{a} = (a^x, a^y)$.

II. CLASSICAL GLUON RADIATION IN ULTRARELATIVISTIC NUCLEAR COLLISIONS

In this section, we briefly summarize the results of [12], as far as they are required for the following. In covariant (Lorentz) gauge, $\partial_\mu A^\mu = 0$, the classical Yang–Mills equations,

$$D_\mu F^{\mu\nu} = J^\nu \;, \tag{1}$$

where $D_\mu \equiv \partial_\mu - ig\,[\,A_\mu\,,\,\cdot\,]\,,\; F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - ig\,[\,A^\mu\,,\,A^\nu\,]$, and J^ν is a classical source current, can be cast into the form

$$\Box A^\mu = J^\mu + ig\,[\,A_\nu\,,\,\partial^\nu A^\mu + F^{\nu\mu}\,]\;, \tag{2}$$

where $\Box$ is the d'Alembertian operator. In this form, the equations can be readily solved perturbatively order by order in the strong coupling constant, g. We expand

$$A_\mu = \sum_{k=0}^{\infty} A_\mu^{(2k+1)} \quad , \quad J_\mu = \sum_{k=0}^{\infty} J_\mu^{(2k+1)} \quad , \tag{3}$$

where $A_\mu^{(n)}$, $J_\mu^{(n)}$ are the contributions of order g^n to the gluon field and source current, respectively. [Eq. (3) takes into account that only odd integers of g occur in this expansion.] Then, the lowest and next-to-lowest order solutions obey

$$\Box A_\mu^{(1)} = J_\mu^{(1)} \equiv \tilde{J}_\mu^{(1)} \;, \tag{4a}$$

$$\Box A_\mu^{(3)} = J_\mu^{(3)} + ig\,[\,A^{(1)\nu}\,,\,\partial_\nu A_\mu^{(1)} + F_{\nu\mu}^{(1)}\,] \equiv \tilde{J}_\mu^{(3)} \;. \tag{4b}$$

These equations are linear to each successive order in g and can therefore be solved with the method of Green functions:

$$A_\mu^{(2k+1)}(x) = \int d^4x'\, G_r(x-x')\,\tilde{J}_\mu^{(2k+1)}(x')\;, \quad k = 0, 1\;. \tag{5}$$

The retarded Green function reads in coordinate and momentum space [14]:

$$G_r(x) = \frac{1}{2\pi}\,\theta(t)\,\delta(x^2) \quad , \quad \tilde{G}_r(k) = -\,\frac{1}{k^2 + i\epsilon k_0} \ . \tag{6}$$

Let us now consider a collision of two nuclei with mass numbers A_1, A_2, moving towards each other with ultrarelativistic velocities, $v_{1,2} \simeq \pm 1$, along the z–axis. The nuclei are taken as ensembles of nucleons [10,12], cf. Fig. 1.

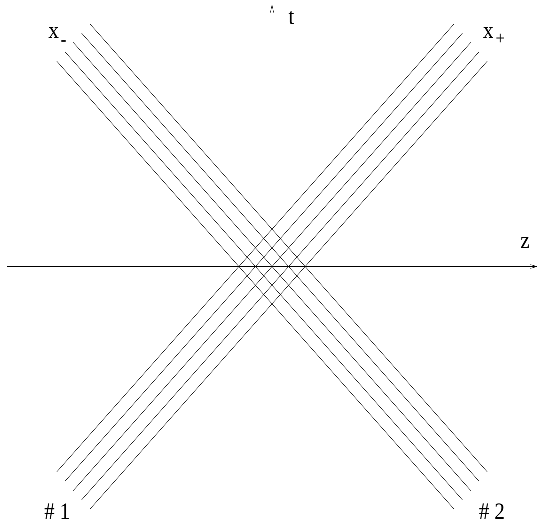

FIG. 1. The nuclear collision as envisaged here.

In order to simplify the color algebra we assume that each "nucleon" consists of a quark–antiquark pair. These valence quarks and antiquarks are confined inside the nucleons (visualized as spheres of equal radius in the rest frame of each nucleus). In order to construct the solution, nucleons inside the nucleus and valence charges inside the nucleons are assumed to be "frozen", i.e., they have definite light-cone (and transverse) coordinates. We assume the charges to move along recoilless trajectories, therefore, their coordinates will not change throughout the calculation. We label the coordinates of the quarks in nucleus 1 by x_{-i}, $\underline{x}_i$, $i = 1, \ldots, A_1$, and those of nucleus 2 by y_{+j}, $\underline{y}_j$, $j = 1, \ldots, A_2$. Antiquark coordinates follow this notation with an additional prime.

Then, the lowest order classical current is a sum of the currents for each individual nucleus, as given in Eq. (3) of Ref. [12]. The lowest order solution, $A^{(1)}_\mu$, and the associated field strength tensor, $F^{(1)}_{\mu\nu}$, were given in Ref. [12], Eqs. (4,5). This solution is identical to the one of the corresponding Abelian problem, i.e., the nuclei pass through each other without interacting. The fields generated by the valence charges simply superpose and no gluons are radiated.

With this lowest order solution and the classical current to next-to-lowest order in g, $J^{(3)}_\mu$, one can compute the next-to-lowest order solution, $A^{(3)}_\mu$. The classical current $J^{(3)}_\mu$ was obtained in [12] from covariant current conservation, $D_\mu\, J^\mu = 0$, and the assumption of recoilless trajectories, see Eq. (8) of [12]. The right-hand side of Eq. (4b) then reads in coordinate space

$$\tilde{J}^{(3)a}_\mu(x) = \sum_{i,j} \left[\tilde{J}^{(3)a}_\mu(x; x_i, y_j) - \tilde{J}^{(3)a}_\mu(x; x'_i, y_j) - \tilde{J}^{(3)a}_\mu(x; x_i, y'_j) + \tilde{J}^{(3)a}_\mu(x; x'_i, y'_j) \right] \ , \tag{7}$$

where

$$\begin{aligned} \tilde{J}^{(3)a}_+(x; x_i, y_j) = \frac{g^3}{(2\pi)^2}\, f^{abc}\, (T^b_i)\, (\tilde{T}^c_j) \Big(&- 2\pi\, \ln(|\underline{x}_i - \underline{y}_j|\lambda)\, \delta(x_- - x_{-i})\, \theta(x_+ - y_{+j})\, \delta(\underline{x} - \underline{x}_i) \\ &+ \ln(|\underline{x} - \underline{x}_i|\lambda)\, \ln(|\underline{x} - \underline{y}_j|\lambda)\, \partial_+ \delta(x_- - x_{-i})\, \delta(x_+ - y_{+j}) \Big) \ , \end{aligned} \tag{8a}$$

$$\tilde{J}_-^{(3)a}(x;x_i,y_j) = \frac{g^3}{(2\pi)^2}\, f^{abc}\, (T_i^b)\, (\tilde{T}_j^c) \Big(2\pi\, \ln(|\underline{x}_i - \underline{y}_j|\lambda)\, \theta(x_- - x_{-i})\, \delta(x_+ - y_{+j})\, \delta(\underline{x} - \underline{y}_j) - \ln(|\underline{x} - \underline{x}_i|\lambda)\, \ln(|\underline{x} - \underline{y}_j|\lambda)\, \delta(x_- - x_{-i})\, \partial_- \delta(x_+ - y_{+j}) \Big) \;, \tag{8b}$$

$$\underline{\tilde{J}}^{(3)a}(x;x_i,y_j) = \frac{g^3}{(2\pi)^2}\, f^{abc}\, (T_i^b)\, (\tilde{T}_j^c)\, \delta(x_- - x_{-i})\, \delta(x_+ - y_{+j}) \times \left(\ln(|\underline{x} - \underline{y}_j|\lambda)\, \frac{\underline{x} - \underline{x}_i}{|\underline{x} - \underline{x}_i|^2} - \ln(|\underline{x} - \underline{x}_i|\lambda)\, \frac{\underline{x} - \underline{y}_j}{|\underline{x} - \underline{y}_j|^2} \right) \;. \tag{8c}$$

Here, (T_i^a), $(\tilde{T}_j^b)$ are color matrices which represent the color charge of the quarks in the color space of nucleon i of nucleus 1 and nucleon j of nucleus 2, while f^{abc} are the structure constants of $SU(N_c)$. The antiquarks have the opposite color charge, $-(T_i^a)$, $-(\tilde{T}_j^b)$, which explains the relative signs in Eq. (7), and ensures color neutrality of each nucleon. λ is an infrared cut-off and acts as gauge parameter for the lowest order solution.

The next-to-lowest order solution is then computed as

$$A_\mu^{(3)a}(x) = -\int \frac{d^4k}{(2\pi)^4}\, \frac{e^{-ik\cdot x}}{k^2 + i\epsilon k_0}\, \tilde{J}_\mu^{(3)a}(k) \;, \tag{9}$$

where

$$\tilde{J}_\mu^{(3)a}(k) = \sum_{i,j} \left[\tilde{J}_\mu^{(3)a}(k;x_i,y_j) - \tilde{J}_\mu^{(3)a}(k;x_i',y_j) - \tilde{J}_\mu^{(3)a}(k;x_i,y_j') + \tilde{J}_\mu^{(3)a}(k;x_i',y_j') \right] \;, \tag{10}$$

and

$$\tilde{J}_+^{(3)a}(k;x_i,y_j) = \frac{g^3}{(2\pi)^2} f^{abc} (T_i^b)\, (\tilde{T}_j^c)\, e^{i(k_+x_{-i} + k_-y_{+j} - \underline{k}\cdot\underline{y}_j)} \int d^2\underline{q}\, e^{-i\underline{q}\cdot(\underline{x}_i - \underline{y}_j)}\, \frac{1}{(\underline{k} - \underline{q})^2} \left[\frac{i}{k_- + i\epsilon} - \frac{ik_+}{\underline{q}^2} \right], \tag{11a}$$

$$\tilde{J}_-^{(3)a}(k;x_i,y_j) = \frac{g^3}{(2\pi)^2} f^{abc} (T_i^b)\, (\tilde{T}_j^c)\, e^{i(k_+x_{-i} + k_-y_{+j} - \underline{k}\cdot\underline{y}_j)} \int d^2\underline{q}\, e^{-i\underline{q}\cdot(\underline{x}_i - \underline{y}_j)}\, \frac{1}{\underline{q}^2} \left[\frac{-i}{k_+ + i\epsilon} + \frac{ik_-}{(\underline{k} - \underline{q})^2} \right], \tag{11b}$$

$$\underline{\tilde{J}}^{(3)a}(k;x_i,y_j) = \frac{g^3}{(2\pi)^2} f^{abc} (T_i^b)\, (\tilde{T}_j^c)\, e^{i(k_+x_{-i} + k_-y_{+j} - \underline{k}\cdot\underline{y}_j)} \int d^2\underline{q}\, e^{-i\underline{q}\cdot(\underline{x}_i - \underline{y}_j)}\, \frac{i(2\underline{q} - \underline{k})}{\underline{q}^2 (\underline{k} - \underline{q})^2} \;. \tag{11c}$$

Note that an explicit expression for the solution $A_\mu^{(3)a}(x)$ was given in Ref. [12], Eqs. (21–24). The field (9) contains a piece associated with the change of color of a charge when it collides with the field of another charge. It arises from the pole $k_\pm = -i\epsilon$ in Eqs. (11a,11b) and thus does not correspond to radiated gluons, cf. also Eq. (25) in [12]. The radiated gluon field, on the other hand, arises from the poles of the retarded propagator in Eq. (9), i.e., it corresponds to on-shell gluons, as one would expect,

$$A_{\mu\,\mathrm{rad}}^{(3)a}(x) = \sum_{i,j} \left[A_{\mu\,\mathrm{rad}}^{(3)a}(x;x_i,y_j) - A_{\mu\,\mathrm{rad}}^{(3)a}(x;x_i',y_j) - A_{\mu\,\mathrm{rad}}^{(3)a}(x;x_i,y_j') + A_{\mu\,\mathrm{rad}}^{(3)a}(x;x_i',y_j') \right] \;, \tag{12}$$

where

$$A_{\mu\,\mathrm{rad}}^{(3)a}(x;x_i,y_j) = \theta(t - t_{ij}) \int d\tilde{k} \left[i\, \tilde{J}_\mu^{(3)a}(\omega,\mathbf{k};x_i,y_j)\; e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} + \mathrm{c.c.} \right] \;, \tag{13}$$

$t_{ij} \equiv (x_{-i} + y_{+j})/\sqrt{2}$ is the time when the collision between the quarks of nucleon i and j happens, and $d\tilde{k} \equiv d^3\mathbf{k}/[(2\pi)^3 2\omega]$, with $\omega = |\mathbf{k}|$. Obviously, this field vanishes prior to the collision.

III. GLUON RADIATION AS A COHERENT STATE

A. Formalism

In the preceding section and Refs. [7,12], the gluons produced in the collision of the two nuclei were treated as a classical field which solves the Yang–Mills equations. In contrast, in this section we want to describe the gluons as quanta radiated in the presence of a classical source, which we will take to be the current (7). The way to solve a problem of this form can be found in standard textbooks [14].

Adapting the usual treatment to our case of a nuclear collision, we note that at $t \to -\infty$ the current (7) is zero and no real gluons are present. The gluonic Fock space "in"–state is therefore the physical vacuum, $|\,0\,\rangle$. When the two nuclei collide, the c–number source current (7) produces real gluons such that at $t \to \infty$, the "out"–state of the system is the *coherent* state [14,15]

$$|\,\tilde{J}\,\rangle \equiv e^{-\bar{N}/2}\,\exp\left[-i\int d\tilde{k}\sum_{\lambda=1}^{2}\sum_{a=1}^{8}\tilde{J}^{(3)a}(\omega,\mathbf{k})\cdot\epsilon(\mathbf{k},\lambda,a)\,\hat{c}^\dagger(\mathbf{k},\lambda,a)\right]|\,0\,\rangle\ , \tag{14}$$

where

$$\tilde{J}^{(3)a}_\mu(\omega,\mathbf{k}) = \int d^4x\, e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}\,\tilde{J}^{(3)a}_\mu(x) \tag{15}$$

is the on-shell Fourier transform of the classical current (7). It is identical with the current (10), taken with on-shell momenta $k^\mu = (\omega,\mathbf{k})$, as implied by our notation. The $\epsilon^\mu(\mathbf{k},\lambda,a)$ are polarization vectors for (real) gluons with 3–momentum $\mathbf{k}$, polarization λ, and color a, obeying

$$k\cdot\epsilon(\mathbf{k},\lambda,a) = 0\ ,\ \ \epsilon(\mathbf{k},\lambda,a)\cdot\epsilon(\mathbf{k},\lambda',a) = g^{\lambda\lambda'}\ . \tag{16}$$

The creation operators $\hat{c}^\dagger(\mathbf{k},\lambda,a)$ and the corresponding annihilation operators $\hat{c}(\mathbf{k},\lambda,a)$ for such gluons fulfill the commutation relation [14]:

$$\left[\hat{c}(\mathbf{k},\lambda,a)\,,\ \hat{c}^\dagger(\mathbf{k}',\lambda',b)\right] = -2\omega\,(2\pi)^3\,g^{\lambda\lambda'}\,\delta_{ab}\,\delta(\mathbf{k}-\mathbf{k}')\ . \tag{17}$$

The first exponential factor in Eq. (14) ensures unitarity. It provides the normalization of the coherent state $|\,\tilde{J}\,\rangle$, with

$$\bar{N} = -\int d\tilde{k}\,\left[\tilde{J}^{(3)a}(\omega,\mathbf{k})\right]^*\cdot\,\tilde{J}^{(3)a}(\omega,\mathbf{k}) \tag{18}$$

being the expectation value of the gluon number (with all possible colors and polarizations) in the coherent state. (The minus sign arises from our choice of metric, $\bar{N}$ is, of course, positive definite.) Since the color quantum number of final states in the collision will not be observed, it is not necessary to decompose the coherent state $|\,\tilde{J}\,\rangle$ in terms of states with good color quantum number [16], i.e., for our purpose it is sufficient to consider the "conventional" coherent state (14).

The coherent state (14) is an eigenstate of the annihilation operator $\hat{c}(\mathbf{k},\lambda,a)$,

$$\hat{c}(\mathbf{k},\lambda,a)\,|\,\tilde{J}\,\rangle = -i\,\tilde{J}^{(3)a}(\omega,\mathbf{k})\cdot\epsilon(\mathbf{k},\lambda,a)\,|\,\tilde{J}\,\rangle\ . \tag{19}$$

Therefore, the expectation value of the number of gluons with momentum $\mathbf{k}$, polarization λ, and color a in the coherent state $|\tilde{J}\,\rangle$ is

$$\langle\,\tilde{J}\,|\,\hat{c}^\dagger(\mathbf{k},\lambda,a)\,\hat{c}(\mathbf{k},\lambda,a)\,|\,\tilde{J}\,\rangle = \left[\tilde{J}^{(3)a}(\omega,\mathbf{k})\right]^*\cdot\epsilon(\mathbf{k},\lambda,a)\,\tilde{J}^{(3)a}(\omega,\mathbf{k})\cdot\epsilon(\mathbf{k},\lambda,a) \tag{20}$$

(no summation over a). Summing over colors and (transverse) polarizations, using

$$\sum_{\lambda=1}^{2}\epsilon^\mu(\mathbf{k},\lambda,a)\,\epsilon^\nu(\mathbf{k},\lambda,a) = -g^{\mu\nu} - \frac{k^\mu k^\nu}{(n\cdot k)^2} + \frac{k^\mu n^\nu + n^\mu k^\nu}{n\cdot k} \tag{21}$$

with an arbitrary time-like unit vector n^μ, and current conservation

$$k \cdot \tilde{J}^{(3)a}(\omega, \mathbf{k}) = 0 \tag{22}$$

[this relation can be readily checked with the explicit form (10,11) for the current $\tilde{J}_\mu^{(3)a}$], we obtain the invariant momentum distribution for the expectation value of the total gluon number in the coherent state $|\, \tilde{J}\, \rangle$,

$$\frac{d\bar{N}}{dy\, d^2\underline{k}} = \frac{1}{2(2\pi)^3} \sum_{\lambda=1}^{2} \sum_{a=1}^{8} \langle\, \tilde{J}\, |\, \hat{c}^\dagger(\mathbf{k}, \lambda, a)\, \hat{c}(\mathbf{k}, \lambda, a)\, |\, \tilde{J}\, \rangle \equiv -\frac{1}{2(2\pi)^3} \left[\tilde{J}^{(3)a}(\omega, \mathbf{k})\right]^* \cdot \tilde{J}^{(3)a}(\omega, \mathbf{k}) \;. \tag{23}$$

Here $y = \frac{1}{2} \ln[k_+/k_-]$ is the (longitudinal) rapidity of the gluons. Again, integrating over invariant momentum space $dy\, d^2\underline{k}$, we obtain the expectation value of the total gluon number $\bar{N}$ in the coherent state $|\, \tilde{J}\, \rangle$, i.e., Eq. (18).

B. Interpretation

The coherent state $|\, \tilde{J}\, \rangle$, Eq. (14), is a superposition of 0–, 1–, 2–, ... , n–, ... gluon states,

$$\begin{aligned} |\, \tilde{J}\, \rangle &= e^{-\bar{N}/2} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \prod_{i=1}^{n} \left[\int d\tilde{k}_i \sum_{\lambda_i=1}^{2} \sum_{a_i=1}^{8} \tilde{J}^{(3)a_i}(\omega_i, \mathbf{k}_i) \cdot \epsilon(\mathbf{k}_i, \lambda_i, a_i)\, \hat{c}^\dagger(\mathbf{k}_i, \lambda_i, a_i) \right] |\, 0\, \rangle \\ &= e^{-\bar{N}/2} \Bigg(|\, 0\, \rangle - i \int d\tilde{k} \sum_{\lambda=1}^{2} \sum_{a=1}^{8} \tilde{J}^{(3)a}(\omega, \mathbf{k}) \cdot \epsilon(\mathbf{k}, \lambda, a)\; |\, \mathbf{k}, \lambda, a\, \rangle \\ &\quad - \frac{1}{2} \int d\tilde{k}\, d\tilde{k}' \sum_{\lambda, \lambda'=1}^{2} \sum_{a,b=1}^{8} \tilde{J}^{(3)a}(\omega, \mathbf{k}) \cdot \epsilon(\mathbf{k}, \lambda, a)\, \tilde{J}^{(3)b}(\omega', \mathbf{k}') \cdot \epsilon(\mathbf{k}', \lambda', b)\; |\, \mathbf{k}, \lambda, a;\, \mathbf{k}', \lambda', b\, \rangle + \ldots \Bigg) \;. \end{aligned} \tag{24}$$

The amplitude for a final state of n gluons with momenta $\mathbf{k}_1, \ldots, \mathbf{k}_n$, polarizations $\lambda_1, \ldots, \lambda_n$, and colors $a_1, \ldots, a_n$ in the coherent state (24) is

$$\begin{aligned} \mathcal{M}(\mathbf{k}_1, \lambda_1, a_1; \ldots; \mathbf{k}_n, \lambda_n, a_n) &\equiv \langle\, \mathbf{k}_1, \lambda_1, a_1; \ldots; \mathbf{k}_n, \lambda_n, a_n\, |\, \tilde{J}\, \rangle \\ &= e^{-\bar{N}/2} \prod_{i=1}^{n} \left[-i\, \tilde{J}^{(3)a_i}(\omega_i, \mathbf{k}_i) \cdot \epsilon(\mathbf{k}_i, \lambda_i, a_i) \right] \;. \end{aligned} \tag{25}$$

Thus, the probability to find n gluons in the final state *regardless* of spin, polarization, and color is

$$P_n = \frac{1}{n!} \int d\tilde{k}_1 \cdots d\tilde{k}_n \sum_{\lambda_1, \ldots, \lambda_n} \sum_{a_1, \ldots, a_n} |\, \mathcal{M}(\mathbf{k}_1, \lambda_1, a_1; \ldots; \mathbf{k}_n, \lambda_n, a_n)\, |^2 \;\;, \tag{26}$$

where the prefactor takes into account that the gluons are indistinguishable Bose particles. A straightforward calculation yields the well-known result

$$P_n = e^{-\bar{N}}\, \frac{\bar{N}^n}{n!} \;\;, \tag{27}$$

where $\bar{N}$ is given by Eq. (18). The emission of gluons follows a Poisson probability distribution, or in other words, it happens in a statistically independent way. Although classically there are $4\, A_1 A_2$ collisions between the quarks and antiquarks, and all these collisions act as classical sources for the gluon field, quantum mechanically there is a certain probability that no gluon, one gluon, ... , n, ... gluons are emitted in the nuclear collision. This probability is given by Eq. (27).

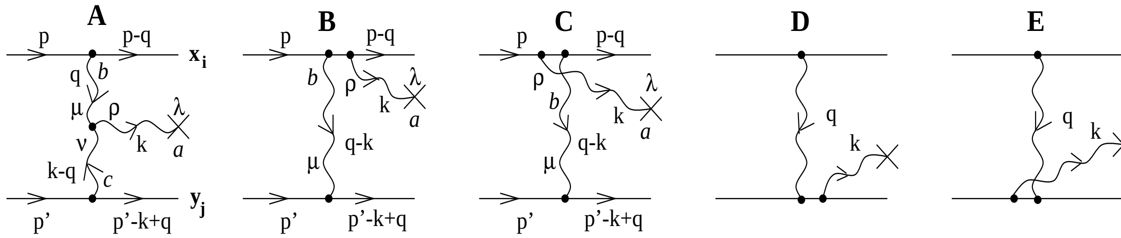

FIG. 2. Lowest order diagrams contributing to the amplitude for the emission of one gluon.

The amplitude (25) for a state where n gluons are emitted during the collision can also be computed diagrammatically with the usual Feynman rules. Let us first focus on the case $n = 1$. To lowest order in g, the amplitude for the emission of one gluon with (on-shell) momentum k^μ, polarization λ, and color a is given by the diagrams in Fig. 2. In the recoilless limit for the quarks, a calculation of these diagrams with the usual Feynman rules yields the result $-i\,\tilde{J}^{(3)a}(\omega, \mathbf{k}) \cdot \epsilon(\mathbf{k}, \lambda, a)$ [Note that in Ref. [12] the gluon field to order g^3, $A_\mu^{(3)a}(x)$, was calculated from the same set of diagrams. The only difference between that calculation and the present one is that now the emitted (on-shell) gluon contributes a factor $\epsilon^\rho(\mathbf{k}, \lambda, a)$ instead of the (retarded) gluon propagator times a phase $e^{-ik\cdot x}$ as in [12], and there is no integration over the gluon 4–momentum k^μ.]

To lowest order in g, the result for the perturbative amplitude for one–gluon emission is therefore identical to the amplitude (25) for $n = 1$. [To lowest order, the prefactor $e^{-\bar{N}/2} \simeq 1$, since $\bar{N} \sim O(g^6)$.] This is another manifestation of the observation made in [8,12] that, to order g^3, classical and quantum calculation give the same result for gluon radiation. Note, however, that although classically there are $4\,A_1 A_2$ collisions with gluon emission, the quantum diagram corresponds to *only one single collision* between any one charge in nucleus 1 and one in nucleus 2 and *one single gluon* emitted as a result of that collision. Of course, there are $4\,A_1 A_2$ possible pairs of colliding charges, corresponding to the $4\,A_1 A_2$ terms in $\tilde{J}_\mu^{(3)a}$, Eq. (10). This increased probability for a collision is reflected in the gluon number distribution which also grows like $4\,A_1 A_2$, cf. Section IV.

To higher order in g, there are the following type of quantum corrections to the one–gluon emission process:

1. loops as shown in Fig. 3a,

2. interactions of quark and antiquark inside a nucleon which participates in the gluon emission process, see for instance Fig. 3b,

3. interactions between quarks and antiquarks of different nucleons in the same nucleus, cf. Fig. 3c, and finally

4. interactions between quarks and antiquarks in different nuclei, Fig. 3d.

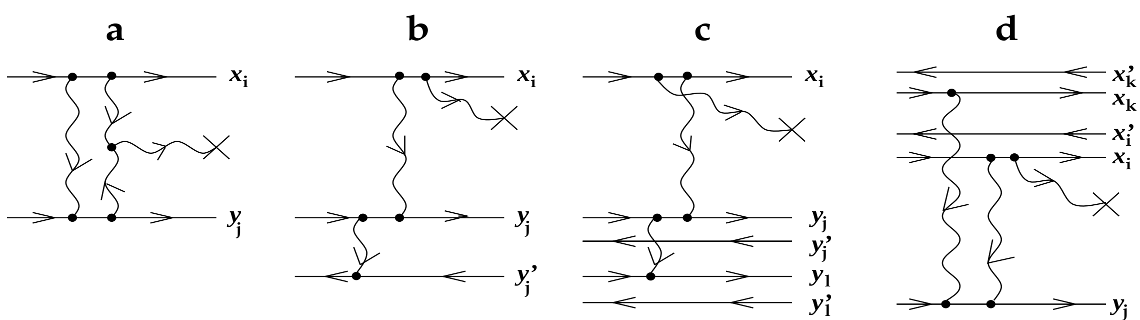

FIG. 3. Higher order quantum corrections to the emission process.

The corrections (2), Fig. 3b, can be absorbed in the nucleon form factor (in the present approach, this form factor will appear after averaging over quark and antiquark coordinates inside a nucleon, cf. Section IV). Moreover, on the time scale of the nuclear collision gluon exchange inside a nucleon can be neglected. Similarly, we argue that corrections of type (3), Fig. 3c, are nuclear structure effects and not important on the scales of interest here. More rigorously, it was shown in [11] that diagrams of such type are exponentially suppressed in the covariant gauge. Corrections (1), Fig. 3a, and (4), Fig. 3d, are higher order quantum effects to the scattering process and suppressed by additional powers of g. Note that the corrections (2–4) induce *correlations* between the two charges directly involved in the gluon emission process and other charges. Following the above arguments, these types of correlations will be neglected.

The amplitudes (25) for states containing n gluons are simply given by *products* of amplitudes for one–gluon states. Each one–gluon amplitude contains $4\,A_1 A_2$ terms. A product of n such amplitudes, corresponding to emission of n gluons, contains $(4\,A_1 A_2)^n$ terms in total. Of these, $(2A_1)!\,(2A_2)!\,/\,(2A_1-n)!\,(2A_2-n)!$ terms correspond to processes where different charges collide *once* (in a given order) and emit a gluon, see e.g. Fig. 4a. These terms are identical to those one would obtain in a perturbative calculation in terms of diagrams, since these diagrams contain no multiple scatterings of a single charge.

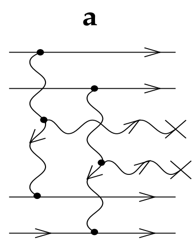

FIG. 4. 2–gluon emission processes, a) from collisions of different charges, b) from collisions where one charge collides twice.

The remaining terms correspond to processes where a given charge collides and emits a gluon *more than once*, cf. Fig. 4b. In the coherent state, such a process is simply given by products of one–gluon amplitudes. Before and after each one–gluon emission process (as represented by the diagrams of Fig. 2), the charge is on-shell. On the other hand, in the corresponding diagrammatic calculation the charge is off-shell. In fact, since the nuclei are highly Lorentz–contracted and since the distance between subsequent collisions is inversely proportional to the corresponding Lorentz factor, the charge is rather strongly off-shell. The coherent state approach as discussed here fails to account for this and thus to give the correct description for the emission of more than one gluon from the same quark line.

In other words, multiple collisions of the same charge induce *correlations* between the n emitted gluons. Such correlations are correctly accounted for in a quantum calculation in terms of diagrams. The coherent state approach, on the other hand, neglects these correlations and assumes that *each* individual gluon is emitted in an *independent* binary collision between color charges. This is what gives rise to the Poisson distribution (27) for the emission of n gluons. Note that in addition to these correlations from multiple collisions, there are also higher order quantum corrections to the n–gluon emission process, which introduce correlations. These correspond to diagrams similar as in Fig. 3. According to the above discussion, these

correlations will be neglected here.

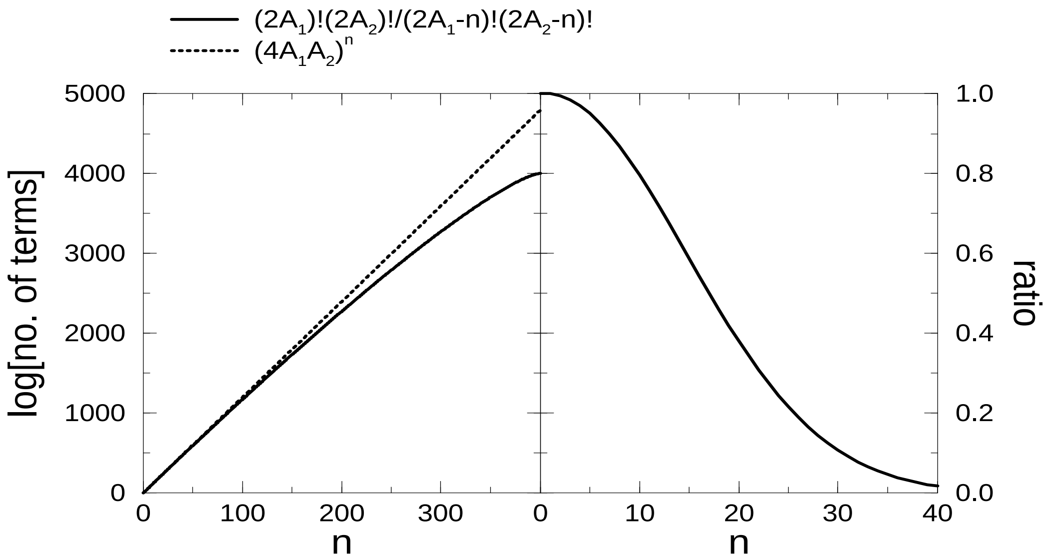

FIG. 5. Left panel: The logarithm of the number of terms corresponding to uncorrelated emission of n gluons (solid) and the logarithm of the total number of terms (dotted) vs. n. The right panel shows the corresponding ratio of these numbers. For the sake of definiteness, $A_1 = A_2 = 200$ was assumed.

In the following we want to derive a sufficient condition under which circumstances gluon emission from independent collisions of different charges dominates that from multiple collisions of the same charge. We argue that if this condition is fulfilled, the coherent state approach is applicable for the description of gluon radiation in ultrarelativistic nuclear collisions. We begin by noting that for small n and large A_1, A_2, the number of terms corresponding to multiple collisions of the same charge is negligible as compared to the number of terms where charges collide only once and emit a gluon. This is graphically shown in Fig. 5 where we plot the latter number (solid line) and the total number of terms (dashed). The right-hand panel shows the ratio of these numbers. Using Stirling's formula for large A_1, A_2, this ratio can be approximated by $\exp[-n(n-1)\,(A_1 + A_2)/4\,A_1 A_2]$, i.e., an approximate Gaussian behavior with n, as indeed seen in Fig. 5. Obviously, for $n(n-1) \ll 4\,A_1 A_2/(A_1 + A_2)$ the number of terms where charges collide only once and emit a gluon dominates the number of terms where gluons are emitted in multiple collisions of the same charge. In the situation of interest to us, $A \equiv A_1 \simeq A_2 \gg 1$, such that for the following discussion we may simplify this condition as $n^2 \ll 2\,A$. This condition is a *sufficient* criterion to neglect gluon emission from multiple collisions of the same charge.

For $n^2 \ll 2\,A$, the n gluons are more likely to be produced in independent collisions of different charges than in multiple collisions of the same charge. In this case, we expect the description in terms of a coherent state to be (approximately) correct. For large $n^2 \sim 2\,A$, on the other hand, the emission of gluons from multiple collisions of the same charge is not negligible. The emission process for these gluons is incorrectly treated in the coherent state approach, and the discrepancy to the correct (quantum) result can in principle become large. Note, however, that a quantitative calculation is required to prove that this is indeed the case. It may be that even for large n^2 the deviation of the coherent state approach from the true answer remains small. Therefore, the criterion $n^2 \ll 2\,A$ is rather conservative.

In order to minimize the deviation of the coherent state approach from the correct result, we have to require that the *probability* of occurrence of quantum states with a large number of gluons (which are incorrectly treated in the coherent state approach) is small. Since that probability follows the Poisson distribution (27) in a coherent state, this happens exactly when the expectation value of the total gluon number in the coherent state, $\bar{N}$, is small. Our criterion then becomes $\bar{N}^2 \ll 2\,A$. Note that if $\bar{N}^2 \ll 2\,A$ is fulfilled, it certainly holds for any subset of the total number $\bar{N}$ of radiated gluons as well. A subset of particular interest to us is, for instance, the number of gluons per rapidity, $d\bar{N}/dy$. Moreover, even if it does not hold for the total gluon number, $\bar{N}$, it may still be satisfied for a smaller subset such as $d\bar{N}/dy$. This is obvious, since gluons in that subset are more likely to be less correlated. In Section V we shall discuss this criterion

in a more quantitative way for the situation of an ultrarelativistic nuclear collision.

Note that this result is at first sight rather surprising. Naively, one would argue that the classical description becomes better when the number of emitted gluons is large, such that the notion of a classical field becomes reasonable. As the description in terms of a coherent state is the closest analogue to the classical approach, one would expect a similar condition to hold in this case, too, for instance, that the expectation value of the number of gluons in the state $|\,\tilde{J}\,\rangle$ should be large. In contrast, from the preceding arguments one concludes that the coherent state approach is more viable for $\bar{N}^2 \ll 2\,A$. This apparent contradiction is resolved noting that *not* a large $\bar{N}$ is the essential criterion but rather that the emission of the n gluons in an n–gluon state happens in an *uncorrelated* fashion. This is exactly what gives rise to the Poisson distribution (27) for the probability to find n gluons in the coherent state. If there are correlations in the emission of the gluons, the probability would no longer be a Poisson distribution, and the coherent state approach is no longer applicable. As discussed above, multiple collisions of the same charge are the major source of correlations and thus for potential deviations from the coherent state approach. The condition $\bar{N}^2 \ll 2\,A$ ensures that such correlations are negligible.

IV. THE EVENT-AVERAGED GLUON NUMBER DISTRIBUTION

The coherent state $|\tilde{J}\,\rangle$ depends on the initial color orientations of the quarks and antiquarks, (T_i^a) , $(\tilde{T}_j^b)$, and their coordinates x_i, x_i', y_j, y_j' in a single collision event. These coordinates and color orientations are supposed to be fixed in the initial state. The coherent state is therefore a pure state. In other words, the phase information about the emitted gluons is complete, and all gluons are emitted *coherently.* This is certainly unphysical, because the emission of two gluons is *incoherent* if the spatial distance between their emission points is larger than their inverse momentum. In the classical approach of Refs. [7,9,10,12] as well as in the coherent state formulation discussed here, this incoherence is introduced *a posteriori* by averaging over all possible color orientations and coordinates. In the following, we therefore compute the respective average of the expectation value of the gluon number distribution (23). We shall refer to this average as *event average*. In an extension of the treatment in [12] we shall now also take a possible finite impact parameter $\underline{b}$ of the nuclear collision into account.

Similar as in [12], we introduce center-of-mass coordinates for nucleon i, $(X_{-i}, \underline{X}_i)$, *as measured in the center of mass of nucleus 1*, and for nucleon j, $(Y_{+j}, \underline{Y}_j)$, *as measured in the center of mass of nucleus 2*, and relative coordinates Δx_{-i}, $\Delta \underline{x}_i$, Δy_{+j}, $\Delta \underline{y}_j$ for the quarks and antiquarks inside each nucleon, as measured from the center of mass of the individual nucleon. Thus, if we assume that the center of mass of nucleus 1 moves along the z–axis with velocity $v_1 = 1$, and the center of mass of nucleus 2 parallel to the z–axis at a transverse distance (impact parameter) $\underline{b}$ and with velocity $v_2 = -1$, then

$$x_{-i} = X_{-i} + \frac{\Delta x_{-i}}{2}\ ,\ \ x'_{-i} = X_{-i} - \frac{\Delta x_{-i}}{2}\ ,\ \ y_{+j} = Y_{+j} + \frac{\Delta y_{+j}}{2}\ ,\ \ y'_{+j} = Y_{+j} - \frac{\Delta y_{+j}}{2}\ , \tag{28a}$$

$$\underline{x}_i = \underline{X}_i + \frac{\Delta \underline{x}_i}{2}\ ,\ \ \underline{x}'_i = \underline{X}_i - \frac{\Delta \underline{x}_i}{2}\ ,\ \ \underline{y}_j = \underline{b} + \underline{Y}_j + \frac{\Delta \underline{y}_j}{2}\ ,\ \ \underline{y}'_j = \underline{b} + \underline{Y}_j - \frac{\Delta \underline{y}_j}{2}\ . \tag{28b}$$

In another generalization of the treatment in [12] we do no longer consider nucleons and nuclei to be "cylindrical". The event average is therefore defined as

$$\begin{aligned}\langle\,\cdot\,\rangle \equiv & \prod_{i=1}^{A_1} \int_1 \frac{d^2\underline{X}_i\, dX_{-i}}{4\pi R_1^3\,/\,3\sqrt{2}\,\gamma} \int_i \frac{d^2(\Delta\underline{x}_i/2)\, d(\Delta x_{-i}/2)}{4\pi a^3\,/\,3\sqrt{2}\,\gamma} \\ & \times \prod_{j=1}^{A_2} \int_2 \frac{d^2\underline{Y}_j\, dY_{+j}}{4\pi R_2^3\,/\,3\sqrt{2}\,\gamma} \int_j \frac{d^2(\Delta\underline{y}_j/2)\, d(\Delta y_{+j}/2)}{4\pi a^3\,/\,3\sqrt{2}\,\gamma}\ \frac{1}{N_c^2}\ \mathrm{tr}\,[\,\cdot\,]\ \ .\end{aligned} \tag{29}$$

Here, R_1, R_2 are the radii of nucleus 1 and 2, respectively, a is the nucleon radius, γ the Lorentz factor in the center-of-mass frame of the nuclei, and the trace is taken over the color space of nucleon i and j. The integrations over "1" and "2" run over the volume of nucleus 1 and nucleus 2, respectively. Integrations over "i" and "j" run over the corresponding volumes of nucleon i and j. All volumes are Lorentz–contracted spheres.

We are now in the position to calculate the *event-averaged* expectation value for the gluon number distribution, Eq. (23). If we follow the same steps as in [12] (and take "cylindrical" nucleons and nuclei) we indeed obtain the same answer, Eq. (39) of [12]. One of the assumptions of that derivation was, however, that the nuclei are (infinitely) large, cf. Eq. (35) of [12]. In that case, the notion of a finite impact parameter is obsolete.

In the following we outline the derivation of the gluon number distribution in the case of a nuclear collision at finite impact parameter $\underline{b}$ and for spherical nucleons and nuclei. Inserting the current (10) into Eq. (23), the expectation value of the gluon number distribution in the coherent state $|\,\tilde{J}\,\rangle$ becomes

$$\frac{d\bar{N}}{dy\,d^2\underline{k}} = 4\,\frac{g^6}{2(2\pi)^7}\,f^{abc}\,f^{ade}\,\frac{1}{\underline{k}^2}\int d^2\underline{q}_1\,d^2\underline{q}_2\,\frac{\underline{q}_1\cdot\underline{q}_2\,\underline{k}^2+\underline{q}_1^2\,\underline{q}_2^2-\underline{q}_1^2\,\underline{k}\cdot\underline{q}_2-\underline{q}_2^2\,\underline{k}\cdot\underline{q}_1}{\underline{q}_1^2\;\underline{q}_2^2\;(\underline{k}-\underline{q}_1)^2\;(\underline{k}-\underline{q}_2)^2} \tag{30}$$
$$\times\sum_{i,k=1}^{A_1}\sum_{j,l=1}^{A_2}(T_i^b)\,(\tilde{T}_j^c)\,(T_k^d)\,(\tilde{T}_l^e)\;\mathcal{P}(k_+,\underline{q}_1;x_i)\,\mathcal{P}(k_-,\underline{k}-\underline{q}_1;y_j)\,\mathcal{P}^*(k_+,\underline{q}_2;x_k)\,\mathcal{P}^*(k_-,\underline{k}-\underline{q}_2;y_l)\ ,$$

where k^μ is on-shell and

$$\mathcal{P}(k_+,\underline{q};x_i)\ \equiv\ e^{ik_+x_{-i}-i\underline{q}\cdot\underline{x}_i}\ -\ e^{ik_+x'_{-i}-i\underline{q}\cdot\underline{x}'_i}\ . \tag{31}$$

We now perform the event average (29). The averaging over color is done utilizing $\mathrm{tr}\,[\,(T_i^b)\,(\tilde{T}_j^c)\,(T_k^d)\,(\tilde{T}_l^e)\,] = \delta_{ik}\,\delta^{bd}\delta_{jl}\,\delta^{ce}/4$, and $f^{abc}\,f^{abc} = N_c(N_c^2-1)$, with the result

$$\frac{d\langle\,\bar{N}\,\rangle}{dy\,d^2\underline{k}} = \frac{g^6}{2(2\pi)^7}\,\frac{N_c^2-1}{N_c}\,\frac{1}{\underline{k}^2}\int d^2\underline{q}_1\,d^2\underline{q}_2\,\frac{\underline{q}_1\cdot\underline{q}_2\,\underline{k}^2+\underline{q}_1^2\,\underline{q}_2^2-\underline{q}_1^2\,\underline{k}\cdot\underline{q}_2-\underline{q}_2^2\,\underline{k}\cdot\underline{q}_1}{\underline{q}_1^2\;\underline{q}_2^2\;(\underline{k}-q_1)^2\;(\underline{k}-q_2)^2}\;\langle\,\tilde{\mathcal{P}}(k,\underline{q}_1,\underline{q}_2)\,\rangle\,, \tag{32}$$

where

$$\tilde{\mathcal{P}}(k,\underline{q}_1,\underline{q}_2) = \sum_{i=1}^{A_1} e^{-i(\underline{q}_1-\underline{q}_2)\cdot\underline{X}_i}\left[\ e^{-i(\underline{q}_1-\underline{q}_2)\cdot\Delta\underline{x}_i/2} - e^{ik_+\Delta x_{-i}-i(\underline{q}_1+\underline{q}_2)\cdot\Delta\underline{x}_i/2} + \text{c.c.}\ \right]$$
$$\times\sum_{j=1}^{A_2} e^{i(\underline{q}_1-\underline{q}_2)\cdot(\underline{b}+\underline{Y}_j)}\left[\ e^{i(\underline{q}_1-\underline{q}_2)\cdot\Delta\underline{y}_j/2} - e^{ik_-\Delta y_{+j}-i\underline{k}\cdot\Delta\underline{y}_j+i(\underline{q}_1+\underline{q}_2)\cdot\Delta\underline{y}_j/2} + \text{c.c.}\ \right]\ . \tag{33}$$

We now average over the light-cone variables Δx_{-i}. The corresponding integration runs between $\pm f/\gamma \equiv \pm\sqrt{a^2-(\Delta\underline{x}_i/2)^2}/\sqrt{2}\gamma$ and gives

$$\int_{-f/\gamma}^{f/\gamma} d\left(\frac{\Delta x_{-i}}{2}\right)\,e^{ik_+\Delta x_{-i}} = \frac{\sin[2\,k_+f/\gamma]}{k_+}\rightarrow\frac{2\,f}{\gamma}\qquad(\gamma\rightarrow\infty)\ . \tag{34}$$

With an analogous relation for the average over Δy_{+j}, the longitudinal momentum dependence in the phase factor drops out. The average over the light-cone variables X_{-i}, Y_{+j} is even simpler, since $\tilde{\mathcal{P}}$ does not depend on them. The integration, however, yields a factor analogous to (34). For the average over transverse coordinates we then use [17]

$$\frac{1}{2\pi\,R^2/3}\int_{|\underline{x}|\leq R} d^2\underline{x}\;e^{i\underline{q}\cdot\underline{x}}\,\sqrt{1-\frac{\underline{x}^2}{R^2}} = \frac{3}{\underline{q}^2R^2}\left[\frac{\sin(|\underline{q}|R)}{|\underline{q}|R}-\cos(|\underline{q}|R)\right] = \frac{3\,j_1(|\underline{q}|R)}{|\underline{q}|R}\equiv\Delta(|\underline{q}|R)\ , \tag{35}$$

where j_1 is a spherical Bessel function [note that for "cylindrical" nucleons and nuclei, $\Delta(x)\equiv 2\,J_1(x)/x$ [12]], with the result

$$\langle\,\tilde{\mathcal{P}}(k,\underline{q}_1,\underline{q}_2)\,\rangle = 4\,A_1A_2\;e^{i(\underline{q}_1-\underline{q}_2)\cdot\underline{b}}\;\Delta(|\underline{q}_1-\underline{q}_2|R_1)\;\Delta(|\underline{q}_1-\underline{q}_2|R_2) \tag{36}$$
$$\times\left[\Delta(|\underline{q}_1-\underline{q}_2|a)-\Delta(|\underline{q}_1+\underline{q}_2|a)\right]\left[\Delta(|\underline{q}_1-\underline{q}_2|a)-\Delta(|2\underline{k}-\underline{q}_1-\underline{q}_2|a)\right]\ .$$

$\Delta(x)$ is a rapidly decreasing function of its argument, thus the factors $\Delta(|\underline{q}_1 - \underline{q}_2|R_i)$, $i = 1, 2$, limit the difference $|\underline{q}_1 - \underline{q}_2|$ to a scale of order $\min\{\pi/R_1, \pi/R_2\}$. That scale is much smaller than the typical scale of variation $\sim 1/a$ of the remaining terms in (36) and in the integral over $\underline{q}_1$, $\underline{q}_2$ in Eq. (32). [An exception is the factor $e^{i(\underline{q}_1-\underline{q}_2)\cdot \underline{b}}$ which, for large impact parameters $|\underline{b}| \sim R_i$, $i = 1, 2$, then varies over the same small scale π/R_i, $i = 1, 2$, and consequently must not be simply approximated by 1.] In these terms, we may therefore take to good approximation $\underline{q}_2 \simeq \underline{q}_1$. [This equality was achieved by quite similar means in [12] via the assumption that if $R_1 > R_2 \gg a$, then $\Delta(|\underline{q}_1 - \underline{q}_2|R_1) \simeq 4\pi/R_1^2\, \delta(\underline{q}_1 - \underline{q}_2)$, cf. Eq. (35) of [12].]

The final result for the event-averaged gluon number distribution reads

$$\frac{d\langle \bar{N} \rangle}{dy\, d^2\underline{k}} = \pi \, \frac{g^6}{(2\pi)^6} \, \frac{N_c^2 - 1}{N_c} \, \langle T_{A_1A_2}(\underline{b}) \rangle \, \frac{4}{\underline{k}^2} \int d^2\underline{q} \, \frac{F(|\underline{q}|a)}{\underline{q}^2} \, \frac{F(|\underline{k} - \underline{q}|a)}{(\underline{k} - \underline{q})^2} \ , \tag{37}$$

with $F(x) = 1 - \Delta(2x)$, and where we defined

$$\langle T_{A_1A_2}(\underline{b}) \rangle \equiv \int \frac{d^2\underline{l}}{(2\pi)^2} \, e^{i\underline{l}\cdot\underline{b}} \, A_1 A_2 \, \Delta(|\underline{l}|R_1)\, \Delta(|\underline{l}|R_2) \ . \tag{38}$$

While Δ in Eq. (37) (cf. the definition of F) stands for the *nucleonic* form factor [13], in (38) it represents the *nuclear* form factor.

We will now show that $\langle T_{A_1A_2}(\underline{b}) \rangle$ is the *event-averaged number of binary nucleon–nucleon collisions per transverse area* in an $A_1 + A_2$–collision at impact parameter $\underline{b}$ (the so-called nuclear *thickness* function). Note that for a central collision, uniform nuclear density distributions, and "cylindrical" nuclei, this factor is $A_1A_2/\pi R_1^2$ (for nucleus 1 being the larger of the two nuclei), and we thus re-obtain the answer Eqs. (38,39) of [12] [up to the modified form factor $F(x)$].

To prove this assertion, we revert the averaging over the center-of-mass coordinates of the nucleons, i.e.,

$$T_{A_1A_2}(\underline{b}) = \int \frac{d^2\underline{l}}{(2\pi)^2} \sum_{i,j} e^{-i\underline{l}\cdot(\underline{X}_i - \underline{Y}_j - \underline{b})} \ , \tag{39}$$

and introduce a factor of

$$1 \equiv \int d^2\underline{X}\, d^2\underline{Y}\, \delta(\underline{X} - \underline{X}_i)\, \delta(\underline{Y} - \underline{Y}_j) \tag{40}$$

under the sum. This enables us to perform the $\underline{l}$–integration. Introducing the baryon number distributions for nucleus 1 and 2,

$$\rho_{A_1}(\underline{X}, Z_1) = \sum_{i=1}^{A_1} \delta(\underline{X} - \underline{X}_i)\, \delta(Z_1 - Z_i) \ , \quad \rho_{A_2}(\underline{Y}, Z_2) = \sum_{j=1}^{A_2} \delta(\underline{Y} - \underline{Y}_j)\, \delta(Z_2 - Z_j) \ , \tag{41}$$

we realize that

$$T_{A_1A_2}(\underline{b}) = \int d^2\underline{X} \int dZ_1\, \rho_{A_1}(\underline{X}, Z_1) \int dZ_2\, \rho_{A_2}(\underline{X} - \underline{b}, Z_2) \tag{42}$$

is *the number of binary nucleon–nucleon collisions per transverse area* for an $A_1 + A_2$–collision at impact parameter $\underline{b}$. The event average of $T_{A_1A_2}(\underline{b})$ is simply

$$\langle T_{A_1A_2}(\underline{b}) \rangle = \int d^2\underline{X} \int dZ_1\, \langle \rho_{A_1}(\underline{X}, Z_1) \rangle \int dZ_2\, \langle \rho_{A_2}(\underline{X} - \underline{b}, Z_2) \rangle \ . \tag{43}$$

For Gaussian nuclear density distributions, $\rho_A(\mathbf{x}) = A\, \exp[-\mathbf{x}^2/R^2]\, (\pi R^2)^{-3/2}$, we obtain the well-known result $\langle T_{A_1A_2}(\underline{b}) \rangle = A_1A_2\, \exp[-\underline{b}^2/(R_1^2 + R_2^2)]/\pi(R_1^2 + R_2^2)$.

We have thus generalized the result of [12] for the gluon number distribution to collisions at finite impact parameter. As expected, the gluon number decreases with the impact parameter according to the decrease of $\langle T_{A_1A_2}(\underline{b}) \rangle$ with $\underline{b}$. We also conclude from the results of this section that the classical solution [12] and the coherent state approach yield the same expression for the gluon number distribution.

V. CONCLUSIONS

In this work we have exhibited the relationship between classical gluon radiation in ultrarelativistic nuclear collisions as calculated in [7,8,12] and corresponding quantum radiation in a coherent state. We have demonstrated that the coherent state formalism yields the same result for the gluon number distribution as the classical approach. A condition for the coherent state approach to represent a valid description of gluon radiation in ultrarelativistic nuclear collisions was found to be that the expectation value of the gluon number in the coherent state has to be small compared to the number of sources in the (classical) source current, since then the emission of individual gluons happens in an uncorrelated fashion. The event-averaged version of this criterion reads $\langle \bar{N} \rangle^2 \ll Z_{\rm eff}$, where $Z_{\rm eff}$ is the effective number of sources in the classical source current, which is, for the simple nuclear model considered here, equal to $2\,A$.

This condition is similar to the criterion $\mu^2 \gg \underline{k}^2 \gg \Lambda^2_{\rm QCD}$ for the applicability of the classical approach of Ref. [9], since the latter states that the (transverse) area density of source charges should be large on transverse momentum scales of interest (such that the source can be described classically) while the number of field quanta $\sim \mu^2/\underline{k}^2$ associated with these sources must not be too large, i.e., it must be small as compared to $\mu^2/\Lambda^2_{\rm QCD}$. The problem is, however, that if one considers valence charges only, such as in the approach advertised here and in Refs. [10,12], in the case of interest, i.e., for instance for a $Au+Au$–collision at RHIC energies, $\sqrt{s} = 200$ AGeV, $\mu^2 = (N_c^2-1)A/(2N_c^2\,\pi R^2)$ [10] is only about $(160\,{\rm MeV})^2$, and thus of the order of $\Lambda^2_{\rm QCD}$ rather than much larger. It was therefore argued [8,18] that besides the valence charges considered here, one should also add quark and gluon charges arising from the nucleonic parton sea.

On the other hand, in order to estimate whether the condition $\langle \bar{N} \rangle^2 \ll 2\,A$ found here in the framework of the coherent state approach is quantitatively fulfilled, we compute the event-averaged gluon number distribution from Eq. (37). [We use the form factor for "cylindrical" nucleons, since in that case an analytical expression for the gluon number distribution is known [12]; the difference between the form factors in the cylindrical and spherical cases is small.] The result is shown in Fig. 6 for a central collision with $A_1 = A_2 = 200$. The strong coupling constant was taken as $\alpha_S \equiv g^2/4\pi = 0.3$, the nucleon radius (entering the nucleonic form factors) was assumed to be $a = 0.8$ fm.

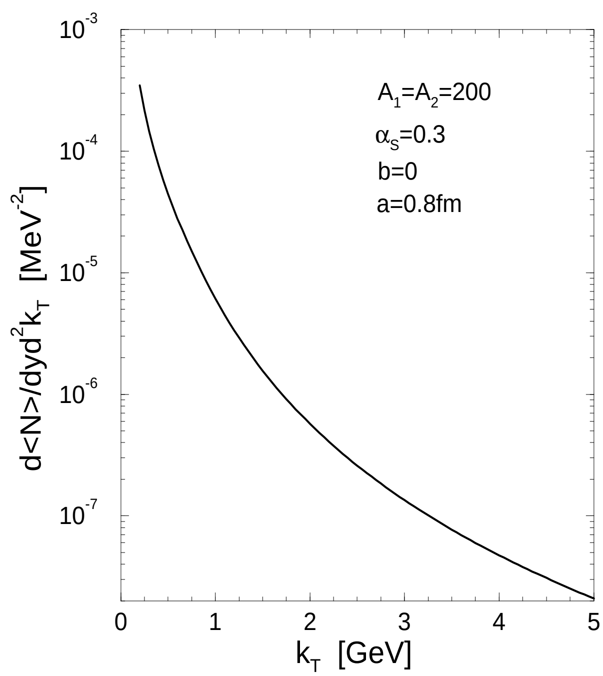

FIG. 6. The gluon number distribution for a central $A_1 + A_2$–collision.

As noted in [12], the infrared behavior of the distribution is $\sim 1/\underline{k}^2$, due to the nucleonic form factors, not $\sim \log(\underline{k}^2)/\underline{k}^4$ as claimed in [7,8]. Therefore, $d\langle \bar{N} \rangle/dy$ diverges only logarithmically. This divergence is the

well-known infrared catastrophe [14]. Physically, it will be regulated on a typical hadronic mass scale due to confinement. Here we simply introduce a lower transverse momentum cut-off $|\underline{k}_{\rm min}|$ to estimate $d\langle \bar{N} \rangle/dy$. Let us first choose the rather low $|\underline{k}_{\rm min}| = \Lambda_{\rm QCD} \simeq 200$ MeV. From a numerical integration of the spectrum in Fig. 6 up to $|\underline{k}| = 10.2$ GeV we obtain for the gluon rapidity density $d\langle \bar{N} \rangle/dy \simeq 140$. The rapidity distribution is constant in y, since the radiating charges are assumed to move along the light-cone. However, since the classical approach fails anyway in the fragmentation regions, $y \sim \pm y_{\rm CM}$ [13], one should not consider gluons other than such produced around midrapidity $y \simeq 0$. We shall therefore use the computed $d\langle \bar{N} \rangle/dy$ (at midrapidity) to check the condition for the viability of the coherent state approach. As discussed in Section III, it is always possible to use a subset for this purpose rather than the total number of gluons. We thus find that $(d\langle \bar{N} \rangle/dy)^2 \simeq 20000 > 2\,A = 400$, i.e., the criterion found in Section III is not satisfied. In other words, the resulting gluon multiplicity is so large that a part of the gluons must have been produced in multiple collisions of the same charge.

On the other hand, gluons with $|\underline{k}| \sim \Lambda_{\rm QCD}$ can certainly no longer be described in perturbative terms. If we consider an even smaller subset and take an infrared cut-off of 1 GeV, we obtain $d\langle \bar{N} \rangle/dy \simeq 26$. For an even larger cut-off of 2 GeV [5], $d\langle \bar{N} \rangle/dy \simeq 8$. For this cut-off, the gluon multiplicity is small enough that these gluons are likely to be produced in a statistically independent way. Thus, the coherent state approach to describe the production of such gluons seems justified.

This result establishes that, although $\mu^2 \sim \Lambda_{\rm QCD}^2$, i.e., in principle violation of the fundamental assumption of the approach of Ref. [9], for practical purposes, i.e., for midrapidity gluons with perturbative transverse momenta $|\underline{k}| > 1 - 2$ GeV and produced in collisions between valence charges, a description in terms of a coherent state generated by a classical source of color charges is likely to be reasonable. The criterion $\mu^2 \gg \underline{k}^2 \gg \Lambda_{\rm QCD}^2$ is certainly a sufficient condition for the applicability of the classical approach. Here we have established that it might actually only be necessary to fulfill $\langle \bar{N} \rangle^2 \ll Z_{\rm eff}$, with $Z_{\rm eff}$ being the effective number of classical sources (and the total number of gluons squared, $\langle \bar{N} \rangle^2$, possibly replaced by a smaller subset of this number). If a quantum calculation of gluon emission from multiple collisions of the same valence charge proves that the deviation from the coherent state result for this process is small, then the range of applicability of the coherent state approach is, for practical purposes, even larger. Note that this is the case also for a smaller coupling constant.

A gluon rapidity density of $d\langle \bar{N} \rangle/dy \sim 140$ or even less for a central, ultrarelativistic collision of $A = 200$–nuclei appears rather small. The reason is that, as already mentioned above, for $A = 200$ the average transverse color charge density squared $\mu^2 = (N_c^2 - 1)A/(2N_c^2\,\pi R^2)$ [10] is only about $(160\,{\rm MeV})^2$. If one adds the charge contained in the nucleonic parton sea [8,18], then $\mu \sim 400$ MeV for collisions at RHIC energies [8], which increases the gluon rapidity density by a factor of 40 (for fixed nuclear radius R). For $|\underline{k}_{\rm min}| = 1$ GeV, this leads to $d\langle \bar{N} \rangle/dy \sim 1000$, for $|\underline{k}_{\rm min}| = 2$ GeV, we obtain $d\langle \bar{N} \rangle/dy \sim 325$. These values are well within in the established range [3,5] of several hundred gluons per unit rapidity, confirming the conclusion of Ref. [8] that the classical description of gluon radiation and the conventional pQCD mini-jet approach give approximately the same results.

One has to note, however, that increasing the effective charge of the classical current by a factor β (when adding the partonic sea) increases the gluon multiplicity by a factor β^2 (for fixed radii of the colliding nuclei). Therefore, the criterion $(d\langle \bar{N} \rangle/dy)^2 \ll Z_{\rm eff}$, where now $Z_{\rm eff}$ is the effective valence *plus* sea charge, may be violated in the process of adding charges from the nucleonic parton sea. In this case, multiple collisions of the same (valence or sea) charge become again important, and a more detailed quantum calculation of these processes is mandatory [19].

Finally, we have generalized the result for the gluon number distribution found in [12] to nuclear collisions at finite impact parameter. As expected [8], the number of binary nucleon–nucleon collisions per transverse area for given impact parameter, $T_{A_1A_2}(\underline{b})$, appears as a prefactor in the final result.

In future applications it is important to incorporate non-Abelian effects on the dynamical evolution of the radiated gluons to study screening, damping, and, eventually, thermalization which leads to the formation of the proposed quark–gluon–plasma state of nuclear matter.

ACKNOWLEDGMENTS

D.H.R. thanks Miklos Gyulassy and Yuri V. Kovchegov for valuable discussions. The authors are indebted to Yuri V. Kovchegov for a critical reading of the manuscript. This work was supported in part by the U.S.

Department of Energy under Contract No. DE-FG02-96ER-40945.

---

[1] B. Müller, Rep. Prog. Phys. **58**, 611 (1995).
[2] J.W. Harris and B. Müller, Annu. Rev. Nucl. Part. Sci. **46**, 71 (1996).
[3] K. Geiger and B. Müller, Nucl. Phys. **B369**, 600 (1992);
K. Geiger, Phys. Rep. **258**, 237 (1995).
[4] K. Kajantie, P.V. Landshoff, and J. Lindfors, Phys. Rev. Lett. **59**, 2527 (1987);
J.P. Blaizot and A.H. Mueller, Nucl. Phys. **B289**, 847 (1987);
K.J. Eskola, K. Kajantie, and J. Lindfors, Nucl. Phys. **B323**, 37 (1989).
[5] X.N. Wang and M. Gyulassy, Phys. Rev. D **44**, 3501 (1991); Phys. Rev. D **45**, 844 (1992); Phys. Rev. Lett. **68**, 1480 (1992); Phys. Lett. B **282**, 466 (1992); Comp. Phys. Comm. **83**, 307 (1994).
[6] K.J. Eskola and M. Gyulassy, Phys. Rev. C **47**, 2329 (1993);
K.J. Eskola, A.V. Leonidov, and P.V. Ruuskanen, Report No. hep-ph/9606406.
[7] A. Kovner, L. McLerran, and H. Weigert, Phys. Rev. D **52**, 6231 (1995); **52**, 3809 (1995).
[8] M. Gyulassy and L. McLerran, Report No. nucl-th/9704034.
[9] L. McLerran and R. Venugopalan, Phys. Rev. D **49**, 2233 (1994); **49**, 3352 (1994); **50**, 2225 (1994).
[10] Yu.V. Kovchegov, Phys. Rev. D **54**, 5463 (1996).
[11] Yu.V. Kovchegov, Phys. Rev. D **55**, 5445 (1997).
[12] Yu.V. Kovchegov and D.H. Rischke, Report No. hep-ph/9704201 (submitted to Phys. Rev. C).
[13] J.F. Gunion and G. Bertsch, Phys. Rev. D **25**, 746 (1982).
[14] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, (McGraw–Hill, New York, 1985).
[15] C. Slotta and U. Heinz, Phys. Lett. B **391**, 469 (1997).
[16] K.-E. Eriksson, N. Mukunda, and B.-S. Skagerstam, Phys. Rev. D **24**, 2615 (1981).
[17] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* , (Academic Press, San Diego, 1980) , Eqs. (6.567.1) and (8.464.3).
[18] J. Jalilian-Marian, A. Kovner, L. McLerran, and H. Weigert, Report No. hep-ph/9606337;
J. Jalilian-Marian, Report No. hep-ph/9609350.
[19] S.G. Matinyan, B. Müller, and D.H. Rischke (in preparation).